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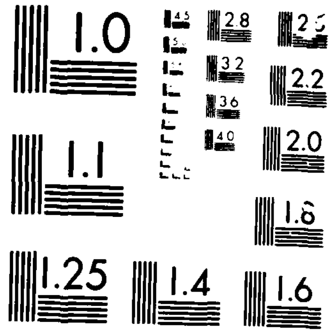

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COMPUTER PROGRAMS FOR ELECTROMAGNETIC SCATTERING FROM A SLOTTED TM CYLINDRICAL CONDUCTOR BY THE PSEUDO-IMAGE METHOD

Interim Technical Report No. 4

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Two computer programs are described and listed. Suppose that an infinitely thin perfectly conducting cylindrical surface with an infinitely long but narrow gap is illuminated by a TM plane wave. The first program uses two different methods, the methods of solution with and without pseudo-image, to calculate the tangential electric field in the gap. The second program uses the Fourier series method of solution to calculate the tangential electric field in the gap for the special case in which the surface is a nearly com-

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20. ABSTRACT (Continue)

plete circular cylindrical shell and the TM plane wave propagates perpendicular to the plane of the gap. To enable the user to verify that they are running correctly, both programs are provided with sample input and output data.

CONTENTS

I. INTRODUCTION

Two methods for calculating electromagnetic plane wave scattering from a perfectly conducting TM cylindrical surface with a narrow but infinitely long slot were described in [1]. These two methods are the method of solution with pseudo-image and the method of solution without pseudo-image. A Fourier series method of solution for calculating the scattering of an electromagnetic plane wave normally incident on a narrow but infinitely long slot in a perfectly conducting TM circular cylindrical surface was also described in [1]. In what follows, two computer programs are described and listed, one that implements the methods of solution with and without pseudo-image, and one that implements the Fourier series method of solution.

The computer program for the methods of solution with and without pseudo-image consists of a main program and the subroutines BES, DECOMP, and SOLVE. The subroutine BES is described and listed in Section II, the subroutines DECOMP and SOLVE in Section III, and the main program in Section IV. The main program calculates the magnetic current coefficient V of [1, eq. (9)] by the methods of solution with and without pseudo-image. The method of solution with pseudo-image is described in [1, Sections III to VII]. The method of solution without pseudo-image is described in [1, Appendix A]. The main program reads input data from the file MAUTZ1.DAT and writes output data on the file MAUTZ2.DAT. In Section IV, the contents of the files MAUTZ1.DAT and MAUTZ2.DAT are listed when the input and output data are for the example of [1, Section VIII].

The computer program for the Fourier series method of solution consists of a main program and the subroutines BES, BESJ1, and BESJY. The subroutine BES is the same as the one mentioned in the previous paragraph. The subroutine BESJ1 is described and listed in Section V, the subroutine BESJY

in Section VI, and the main program in Section VII. The main program obtains an approximation for the magnetic current coefficient $\vec{V}$ of [1, eq. (B-25)] by truncating the infinite series in [1, eq. (B-25)] at $n = N2$ where N2 is entered as input data. The main program reads input data from the file MAUTZ3.DAT and writes output data on the file MAUTZ4.DAT. In Section VII, the contents of the files MAUTZ3.DAT and MAUTZ4.DAT are listed when the input and output data are for the example of [1, Section VIII] and N2 = 10,000.

II. THE SUBROUTINE BES

The subroutine BES(N,X) puts

$$\left.\begin{array}{l} J_n(X) \text{ in } BJ(n+1) \\ Y_n(X) \text{ in } BY(n+1) \end{array}\right\} , \quad n = 0, 1,2,\ldots,N \qquad (1)$$

where J_n is the Bessel function of the first kind of order n, and Y_n is the Bessel function of the second kind of order n. Moreover, N is a non-negative integer and X is a non-negative real number. The subroutine BES is listed at the end of this section. In the COMMON statement on line 3 of this listing, YO, PI2, PI4, and PI7 are input variables and BJ and BY are the output variables that appear in (1). The elements of YO will be described later. Their values should not deeply concern the user because he will never have to change them. The values of the input variables YO(1), YO(2),...,YO(33) are listed as

```
-0.3072582E+04 0.7368758E+04-0.6085100E+03 0.1710234E+02-0.2271001E+00
 0.1600171E-02-0.5961089E-05 0.9545773E-08 0.4163150E+05 0.3420211E+03
 0.1000000E+01-0.6024727E+04 0.1613512E+04-0.7532210E+02 0.1402590E+01
-0.1275602E-01 0.5832787E-04-0.1107698E-06 0.3072946E+05 0.2886431E+03
 0.1000000E+01 0.9999999E+00-0.1097659E-02 0.2461455E-04 0.1000000E+01
 0.1829893E-02-0.3191328E-04-0.1562498E-01 0.1427079E-03-0.5937434E-05
 0.4687498E-01-0.1998720E-03 0.7317495E-05
```

where the jth number in the ith row is YO(5(i-1)+j) for j=1,2,...,5 and i=1,2,3,...,7. The values of the remaining input variables in the common statement are given by

$$\left.\begin{aligned} PI2 &= 2/\pi \\ PI4 &= \pi/4 \\ PI7 &= 3\pi/4 \end{aligned}\right\} \tag{2}$$

Minimum allocations are given by

```
COMMON BJ(N+1), BY(N+1)
DIMENSION AJ(11+N+2[X])
```

where [X] is the largest integer that does not exceed X.

The Bessel functions $\{J_n(X),\ n=0,1,2,\ldots,N\}$ are calculated in lines 5 to 23. As suggested in [2, Sec. 9.12., Example 1], $J_{MZ}(X)$ and $J_{MZ-1}(X)$ are set equal to the arbitrary values of zero and 10^{-20}, respectively, where

$$MZ = 10 + N + 2[X] \tag{3}$$

This is done in lines 5 to 8 where $J_{MZ}(X)$ and $J_{MZ-1}(X)$ are stored in AJ(MZ+1) and AJ(MZ), respectively. In DO loop 16, the recurrence relation [2, eq. (9.1.27)]

$$J_n(X) = \frac{2(n+1)}{X} J_{n+1}(X) - J_{n+2}(X) \tag{4}$$

is used to calculate $\{J_n(X),\ n = MZ-2, MZ-3,\ldots,0\}$. Line 13 puts $J_{MK-1}(X)$ of (4) in AJ(MK). According to [2, eq.(9.1.46)], the calculated values of $\{J_n(X)\}$ have to be normalized by dividing by α where

$$\alpha = J_o(X) + 2J_2(X) + 2J_4(X) + \ldots \tag{5}$$

Each of the calculated values of $\{J_n(X),\ n=0,1,2,\ldots,N\}$ is divided by α in DO loop 17.

The Bessel functions $\{Y_n(X),\ n=0,1,2,\ldots,N\}$ are calculated in lines 24 to 68. For $X \leq 8$, we let [3, eq. (6.8.27)]

$$Y_o(X) = \bar{Y}_o(X) + \frac{2}{\pi} J_o(X)\ \ln(X) \tag{6}$$

where

$$\bar{Y}_o(X) = \frac{\sum_{m=0}^{7} (P0m)X^{2m}}{\sum_{m=0}^{2} (Q0m)X^{2m}} \tag{7}$$

where $\{P0m\}$ and $\{Q0m\}$ are the coefficients given under the heading YZERO 6234 on page 309 of [3]. For $X \leq 8$, we also let [3, eq. (6.8.28)]

$$Y_1(X) = \bar{Y}_1(X) + \frac{2}{\pi} [J_1(X)\ \ln(X) - 1/X] \tag{8}$$

where

$$\frac{\bar{Y}_1(X)}{X} = \frac{\sum_{m=0}^{6} (P0m)X^{2m}}{\sum_{m=0}^{2} (Q0m)X^{2m}} \tag{9}$$

where $\{P0m\}$ and $\{Q0m\}$ are the coefficients given under the heading YONE 6434 on page 315 of [3]. The coefficients $\{P0m\}$ of (7), $\{Q0m\}$ of (7), $\{P0m\}$ of (9), and $\{Q0m\}$ of (9) reside in YO(1) to YO(8), YO(9) to YO(11), YO(12) to YO(18), and YO(19) to YO(21), respectively. Lines 41 and 42 put $\bar{Y}_o(X)$ of (7) in S, and line 44 puts $Y_o(X)$ of (6) in BY(1). Lines 46 and 47 put $\bar{Y}_1(X)/X$ of (9) in S, and line 48 puts $Y_1(X)$ of (8) in BY(2).

For $X > 8$, we let [3, eq. (6.8.17)]

$$Y_n(X) = \sqrt{\frac{2}{\pi X}}\ \{P_n(X)\ \sin(X_n) + Q_n(X) \cos(X_n)\}, \quad n=0,1 \tag{10}$$

In (10),

$$X_n = X - \frac{(2n+1)}{4} \qquad (11)$$

and

$$P_o(X) = \sum_{m=0}^{2} (P0m)Z^{2m} \qquad (12)$$

where

$$Z = 8/X \qquad (13)$$

and {P0m} are the coefficients given under the heading PZERO 6501 on page 320 of [3]. Moreover,

$$Q_o(X) = Z \sum_{m=0}^{2} (P0m)Z^{2m} \qquad (14)$$

where {P0m} are the coefficients given under the heading QZERO 6900 on page 327 of [3]. Regarding (14), there are errors on page 149 of [3]. On that page, each $Q_o(x)/x$ should be replaced by $Q_o(x)/z$ and each $Q_1(x)/x$ by $Q_1(x)/z$. In (10),

$$P_1(X) = \sum_{m=0}^{2} (P0m)Z^{2m} \qquad (15)$$

where {P0m} are the coefficients given under the heading PONE 6701 on page 323 of [3], and

$$Q_1(X) = Z \sum_{m=0}^{2} (P0m)Z^{2m} \qquad (16)$$

where {P0m} are the coefficients given under the heading QONE 7101 on page 331 of [3]. The coefficients {P0m} of (12), {P0m} of (15), {P0m} of (14), and {P0m} of (16) reside in Y0(22) to Y0(24), Y0(25) to Y0(27), Y0(28) to Y0(30), and Y0(31) to Y0(33), respectively. Line 54 puts $P_o(X)$ of (12) in S1, line 55 puts $Q_o(X)$ of (14) in S3, line 56 puts X_o of (11) in S5, and line 57 puts $Y_o(X)$ of (10) in BY(1). Line 59 puts $P_1(X)$ of (15) in S2, line 60 puts $Q_1(X)$ of (16) in S4, line 61 puts X_1 of (11) in S6, and line 62 puts $Y_1(X)$ of (10) in BY(2).

If N > 1, then DO loop 21 uses the recurrence relation [2, eq. (9.1.27)]

$$Y_n(X) = \frac{2(n-1)}{X} Y_{n-1}(X) - Y_{n-2}(X) \tag{17}$$

to put $Y_n(X)$ in BY(n+1) for n=2,3,...,N.

```
C         LISTING OF THE SUBROUTINE BES
          SUBROUTINE BES(N,X)
          COMMON Y0(33),PI2,PI4,PI7,BJ(100),BY(100)
          DIMENSION AJ(100)
          MZ=10+N+2*IFIX(X)
          IF(X.LT.1.E-3) MZ=4+N
          AJ(MZ+1)=0.
          AJ(MZ)=1.E-20
          M1=MZ-1
          X2=2./X
          DO 16 K=1,M1
          MK=MZ-K
          AJ(MK)=X2*FLOAT(MK)*AJ(MK+1)-AJ(MK+2)
 16       CONTINUE
          ALP=.5*AJ(1)
          DO 15 J=3,MZ,2
          ALP=ALP+AJ(J)
 15       CONTINUE
          ALP=2.*ALP
          NP=N+1
          DO 17 K=1,NP
          BJ(K)=AJ(K)/ALP
 17       CONTINUE
```

```
      IF(X-8.) 18,18,19
18    Z1=X*X
      Z2=0.
      Z3=0.
      Z4=0.
      Z5=0.
      Z6=0.
      Z7=0.
      IF(Z1.LT.1.E-11) GO TO 22
      Z2=Z1*Z1
      Z3=Z2*Z1
      IF(Z1.LT.1.E-6) GO TO 22
      Z4=Z3*Z1
      Z5=Z4*Z1
      IF(Z1.LT.1.E-4) GO TO 22
      Z6=Z5*Z1
      Z7=Z6*Z1
22    S=(YO(1)+YO(2)*Z1+YO(3)*Z2+YO(4)*Z3+YO(5)*Z4+YO(6)*Z5+
     1 YO(7)*Z6+YO(8)*Z7)/(YO(9)+YO(10)*Z1+YO(11)*Z2)
      TLG=ALOG(X)
      BY(1)=S+PI2*BJ(1)*TLG
      IF(N.LE.0) RETURN
      S=(YO(12)+YO(13)*Z1+YO(14)*Z2+YO(15)*Z3+YO(16)*Z4+
     1 YO(17)*Z5+YO(18)*Z6)/(YO(19)+YO(20)*Z1+YO(21)*Z2)
      BY(2)=X*S+PI2*(BJ(2)*TLG-1./X)
      GO TO 20
19    Z=8./X
      Z1=Z*Z
      Z2=Z1*Z1
      S=SQRT(PI2/X)
      S1=YO(22)+YO(23)*Z1+YO(24)*Z2
      S3=Z*(YO(28)+YO(29)*Z1+YO(30)*Z2)
      S5=X-PI4
      BY(1)=S*(S1*SIN(S5)+S3*COS(S5))
      IF(N.LE.0) RETURN
      S2=YO(25)+YO(26)*Z1+YO(27)*Z2
      S4=Z*(YO(31)+YO(32)*Z1+YO(33)*Z2)
      S6=X-PI7
      BY(2)=S*(S2*SIN(S6)+S4*COS(S6))
20    IF(NP.LE.2) RETURN
      DO 21 K=3,NP
      K2=K-2
      BY(K)=X2*FLOAT(K2)*BY(K-1)-BY(K2)
21    CONTINUE
      RETURN
      END
```

III. THE SUBROUTINES DECOMP AND SOLVE

The subroutines DECOMP(N, IPS, UL) and SOLVE(N, IPS, UL, B, X) solve a system of linear equations in N unknowns. The input to DECOMP consists of N and the N by N matrix of coefficients on the left-hand side of the matrix equation stored by columns in UL. The output from DECOMP is IPS and UL. This output is fed into SOLVE. The rest of the input to SOLVE consists of N and the column of coefficients on the right-hand side of the matrix equation stored in B. SOLVE puts the solution to the matrix equation in X.

Minimum allocations are given by

```
COMPLEX UL(N*N)
DIMENSION SCL(N), IPS(N)
```

in DECOMP and by

```
COMPLEX UL(N*N), B(N), X(N)
DIMENSION IPS(N)
```

in SOLVE.

More detail concerning DECOMP and SOLVE is on pages 46-49 of [4].

```
C       LISTING OF THE SUBROUTINE DECOMP
        SUBROUTINE DECOMP(N,IPS,UL)
        COMPLEX UL(1600),PIVOT,EM
        DIMENSION SCL(40),IPS(40)
        DO 5 I=1,N
        IPS(I)=I
        RN=0.
        J1=I
        DO 2 J=1,N
        ULM=ABS(REAL(UL(J1)))+ABS(AIMAG(UL(J1)))
        J1=J1+N
        IF(RN-ULM) 1,2,2
1       RN=ULM
2       CONTINUE
        SCL(I)=1./RN
5       CONTINUE
        NM1=N-1
        K2=0
        DO 17 K=1,NM1
        BIG=0.
        DO 11 I=K,N
        IP=IPS(I)
        IPK=IP+K2
        SIZE=(ABS(REAL(UL(IPK)))+ABS(AIMAG(UL(IPK))))*SCL(IP)
        IF(SIZE-BIG) 11,11,10
10      BIG=SIZE
        IPV=I
11      CONTINUE
        IF(IPV-K) 14,15,14
14      J=IPS(K)
        IPS(K)=IPS(IPV)
        IPS(IPV)=J
15      KPP=IPS(K)+K2
        PIVOT=UL(KPP)
        KP1=K+1
        DO 16 I=KP1,N
        KP=KPP
        IP=IPS(I)+K2
        EM=-UL(IP)/PIVOT
18      UL(IP)=-EM
        DO 16 J=KP1,N
        IP=IP+N
        KP=KP+N
        UL(IP)=UL(IP)+EM*UL(KP)
16      CONTINUE
        K2=K2+N
17      CONTINUE
        RETURN
        END
```

```
C     LISTING OF THE SUBROUTINE SOLVE
      SUBROUTINE SOLVE(N,IPS,UL,B,X)
      COMPLEX UL(1600),B(40),X(40),SUM
      DIMENSION IPS(40)
      NP1=N+1
      IP=IPS(1)
      X(1)=B(IP)
      DO 2 I=2,N
      IP=IPS(I)
      IPB=IP
      IM1=I-1
      SUM=0.
      DO 1 J=1,IM1
      SUM=SUM+UL(IP)*X(J)
    1 IP=IP+N
    2 X(I)=B(IPB)-SUM
      K2=N*(N-1)
      IP=IPS(N)+K2
      X(N)=X(N)/UL(IP)
      DO 4 IBACK=2,N
      I=NP1-IBACK
      K2=K2-N
      IPI=IPS(I)+K2
      IP1=I+1
      SUM=0.
      IP=IPI
      DO 3 J=IP1,N
      IP=IP+N
    3 SUM=SUM+UL(IP)*X(J)
    4 X(I)=(X(I)-SUM)/UL(IPI)
      RETURN
      END
```

IV. THE MAIN PROGRAM FOR THE METHODS OF SOLUTION WITH AND WITHOUT PSEUDO-IMAGE

The main program for the methods of solution with and without pseudo-image calculates the magnetic current coefficient V of [1, eq. (9)] by means of the methods of solution with and without pseudo-image. The method of solution with pseudo-image is described in [1, Sections III to VII]. The method of solution without pseudo-image is described in [1, Appendix A]. Input data are read from the file MAUTZ1.DAT, output data are written on the file MAUTZ2.DAT, and the subroutines BES, DECOMP, and SOLVE are called. The subroutine BES was described and listed in Section II and the subroutines DECOMP and SOLVE in Section III.

The main program for the methods of solution with and without pseudo-image is listed at the end of this section. In the open statements on lines 9 and 10 of this listing, MAUTZ1.DAT is the input data file and MAUTZ2.DAT is the output data file. The input data are read early in the program according to

```
      READ(20, 11) N, NG, BK, PINC
11    FORMAT(2I3, 2E14.7)
      READ(20, 13)(X(I), I = 1,N)
13    FORMAT(5E14.7)
      READ(20, 13)(Y(I), I = 1,N)
      READ(20, 13)(YO(I), I = 1, 33)
      READ(20, 13)(XG(I), I = 1, NG)
      READ(20, 13)(AG(I), I = 1, NG)
```

Here, N is the integer that appears in [1, eq. (22)]. That is, N is the number of electric current expansion functions on the complete conducting

surface S^{sc}. An NG-point Gaussian quadrature formula is used whenever Gaussian quadrature is called for. Specifically, the integral from -1 to 1 with respect to x of a function f(x) is approximated by [5, Appendix A]

$$\int_{-1}^{1} f(x)dx = \sum_{\ell=1}^{NG} A_\ell^{(NG)} f(X_\ell^{(NG)}) \tag{18}$$

where $X_\ell^{(NG)}$ and $A_\ell^{(NG)}$ are, respectively, the abscissa and weight given in [5, Appendix A]. Still in the first read statement, BK is the wave number k that appears in [1, eq. (44)], and PINC is the incident angle ϕ^{inc} in [1, eq. (44)]. In the next two read statements, (X(I), Y(I)) are the (x,y) coordinates of the point P_I in [1, Fig. 1]. That is, X(I) and Y(I) are, respectively, the quantities x_I and y_I in [1, eq. (53)]. Here, BK is in reciprocal meters, PINC is in radians, and (X(I), Y(I)) are in meters. The array YO contains input data for the subroutine BES. The values of the elements of YO were given in Section II. These values should not deeply concern the user because he will never have to change them. In the last two read statements, XG(I) and AG(I) are, respectively, the abscissa $X_I^{(NG)}$ and weight $A_I^{(NG)}$ in (18).

Minimum allocations are given by

```
COMMON BJ(2), BY(2)
COMPLEX Z(N*N), VA(N), VINC(N), CURI(N), CURA(N)
DIMENSION X(N), Y(N), XG(NG), AG(NG), DX(N)
DIMENSION DY(N), G(N), XP(N), YP(N), IPS(N)
```

In the present program, BJ(2) and BY(2) are sufficient. However, BJ(100) and BY(100) were used in the listing of this program. The reason for this is explained in the next two sentences. The allocations for BJ and BY in

the present program must be the same as the allocations in the subroutine BES. The same subroutine BES was also used with the program of Section VII, and during that usage BJ(2) and BY(2) did not suffice.

Immediately after the main program at the end of this section, the contents of the input data file MAUTZ1.DAT and the output data file MAUTZ2.DAT are listed when the input and output data are for the example of [1, Section VIII]. The output data file MAUTZ2.DAT contains all the data put out by the write statements in the main program for the methods of solution with and without pseudo-image. The contents of the output data file MAUTZ2.DAT are described in the next two paragraphs.

The input data are written out immediately after they are read in. The four numbers in the ith line under the heading "VINC" are, from left to right, $\mathrm{Re}(4kV^{inc}_{2i-1})$, $\mathrm{Im}(4kV^{inc}_{2i-1})$, $\mathrm{Re}(4kV^{inc}_{2i})$, and $\mathrm{Im}(4kV^{inc}_{2i})$ where V^{inc}_{2i-1} and V^{inc}_{2i} are the (2i-1)th and (2i)th elements of the column vector $\vec{V}^{inc}$ that appears in [1, eq. (23)]. Moreover, "Re" denotes real part, "Im" denotes imaginary part, and k is the wave number that appears in [1, eq. (44)]. Similarly, the numbers written under the heading "VA" are the elements of $4k\vec{V}^{a}$, those written under the heading "Z" are the elements of $\frac{4k}{\eta} Z_{,1}$, those written under the heading "CURI" are the elements of $\eta\vec{I}^{inc}$, and those written under the heading "CURA" are the elements of $\eta\vec{I}^{a}$. Here, $\vec{V}^{a}$ appears in [1, eq. (34)], $Z_{,1}$ denotes the first column of the square matrix Z that appears in [1, eq. (23)], η is the intrinsic impedance that appears in [1, eq. (45)], $\vec{I}^{inc}$ appears in [1, eq. (23)], and $\vec{I}^{a}$ appears in [1, eq. (34)].

Other output variables are called CA, CP, CINC, YHS, YAB, TI, V, YABW, and VW. They are conspicuously identified in the output data file MAUTZ2.DAT. For instance, the real and imaginary parts of CA are preceded

by "CA=". The output variable CINC, being real, is a single number preceded by "CINC=". Each output variable mentioned in the first sentence of this paragraph corresponds to a quantity in [1]. This correspondence is revealed in Table 1.

Table 1. Computer program output variables and their corresponding quantities in [1] or in the text.

Output variable	Corresponding quantity in [1] or in the text	Equation(s) where the quantity appears
N	N	[1, eq. (22)]
NG	NG	(18)
BK	k	[1, eq. (44)]
PINC	ϕ^{inc}	[1, eq. (44)]
X(I)	x_I	[1, eq. (53)]
Y(I)	y_I	[1, eq. (53)]
YO	POm and QOm	(7),(9),(12),(15),(14),(16)
XG(I)	$X_I^{(NG)}$	(18)
AG(I)	$A_I^{(NG)}$	(18)
VINC(I)	$4kV_I^{inc}$	[1, eq. (23)]
VA(I)	$4kV_I^a$	[1, eq. (34)]
Z(I)	$\frac{4k}{\eta} Z_{I1}$	[1, eq. (23)]
CA	C_A	[1, eq. (86)]
CP	C_ϕ	[1, eq. (86)]
CINC	$k\eta C^{inc}$	[1, eq. (116)]
CURI(I)	ηI_I^{inc}	[1, eq. (23)]
CURA(I)	ηI_I^a	[1, eq. (34)]
YHS	$2k\eta Y^{hs}$	[1, eq. (76)]
YAB	$4k\eta(Y^a + Y^b)$	[1, eq. (10)]
TI	$4k\eta I$	[1, eq. (10)]
V	V	[1, eq. (10)]
YABW	$4k\eta(Y^{aw} + Y^{bw})$	[1, eq. (A-25)]
VW	V^w	[1, eq. (A-26)]

Lines 32 and 33 put $\cos\phi^{inc}$ and $\sin\phi^{inc}$ of [1, eq. (44)] in CSP and SNP, respectively. Lines 35 to 37 define the common variables (2) that are input data for the subroutine BES. With regard to [1, eqs. (57) and (58)], DO loop 37 puts $0.5k(x_{J+1} - x_J)$ in DX(J) and $0.5k(y_{J+1} - y_J)$ in DY(J). DO loop 37 also puts $0.5\gamma_J$ in G(J), kx_J^+ in XP(J), and ky_J^+ in YP(J) where γ_J, x_J^+, and y_J^+ are given by [1, eqs. (53), (55), and (56)].

In nested DO loops 38 and 39, $\frac{4k}{\eta} Z_{IJ}$ is put in Z((J-1)*N+I) where Z_{IJ} is given by [1, eq. (52)]. Thus, as the indices I and J traverse their ranges, the upper triangular portion of the square matrix $\frac{4k}{\eta} Z$ of [1, eq. (23)] is stored by columns in the singly dimensioned array Z. When J=N, DO loop 39 puts $4kV_I^{inc}$ in VINC(I) and $4kV_I^a$ in VA(I) where V_I^{inc} and V_I^a are given by [1, eqs. (67) and (74)].

The calculation of the off-diagonal elements of $\frac{4k}{\eta} Z$ of [1, eq. (23)] is described in this paragraph and the next two paragraphs. Replacing ij by IJ in [1, eq. (52)], multiplying both sides of that equation by $4k/\eta$, changing the variables of integration therein to make the ranges of integration from -1 to 1, and then approximating both integrals by means of (18), we obtain

$$\frac{4k}{\eta} Z_{IJ} = \frac{\gamma_I \gamma_J}{4} \sum_{K=1}^{NG} A_K^{(NG)} \sum_{L=1}^{NG} A_L^{(NG)} H_o^{(2)}(\alpha_{IJKL}) \tag{19}$$

where

$$\alpha_{IJKL} = \sqrt{(XX)^2 + (YY)^2} \tag{20}$$

where

$$XX = kx_I^+ - kx_J^+ + \frac{k}{2}(x_{I+1} - x_I)X_K^{(NG)} - \frac{k}{2}(x_{J+1} - x_J)X_L^{(NG)} \tag{21}$$

and

$$YY = ky_I^+ - ky_J^+ + \frac{k}{2}(y_{I+1} - y_I)X_K^{(NG)} - \frac{k}{2}(y_{J+1} - y_J)X_L^{(NG)} \tag{22}$$

Expressions (21) and (22) were obtained by using [1, eqs. (57) and (58)] to express the trigonometric functions in [1, eq. (54)].

Nested DO loops 40 and 41 perform the double summation in (19). The DO loop indices K and L obtain, respectively, the summation indices K and L in (19). DO loop 41 accumulates in Z2 the summation with respect to L in (19). DO loop 40 accumulates in Z1 the summation with respect to K in (19). Line 80 puts α_{IJKL} of (20) in X1. Depending on whether J = N, either line 82 or line 84 puts the Bessel functions $J_o(\alpha_{IJKL})$ and $Y_o(\alpha_{IJKL})$ in BJ(1) and BY(1), respectively. According to [2, eq. (9.1.4)],

$$H_n^{(2)}(x) = J_n(x) - jY_n(x) \tag{23}$$

Line 86 uses (23) with n=0 and $x=\alpha_{IJKL}$ to properly increment Z2. Line 93 puts the right-hand side of (19) in Z((J-1)*N+I).

If, in (19), I and J are interchanged, and the order of summation is interchanged, the right-hand side of (19) remains unchanged. Therefore, Z_{IJ} of (19) is symmetric, that is,

$$Z_{JI} = Z_{IJ} \tag{24}$$

Using (24), line 102 puts $\frac{4k}{\eta} Z_{JI}$ of (19) in Z((I-1)*N+J). Thus, as I and J traverse their ranges, line 102 stores by columns the lower triangular portion of the square matrix $\frac{4k}{\eta}$ Z of [1, eq. (23)] in the singly dimensioned array Z.

Multiplying both sides of [1, eq. (67)] by 4k, replacing i by I therein, expanding cos $(\phi_I - \phi^{inc})$, and using [1, eqs. (57) and (58)] to express the resulting cos ϕ_I and sin ϕ_I terms, we obtain

$$4kV_I^{inc} = 4\gamma_I \, e^{jk(x_I^+ \cos \phi^{inc} + y_I^+ \sin \phi^{inc})} \left(\frac{\sin (S1)}{S1}\right) \tag{25}$$

where

$$S1 = 0.5k \, (x_{I+1} - x_I) \cos \phi^{inc} + 0.5k(y_{I+1} - y_I) \sin \phi^{inc} \tag{26}$$

Equation (25) is written with the understanding that sin(S1)/S1 should be replaced by 1 whenever S1 = 0. If J = N, then line 96 inside nested DO loops 38 and 39 puts the right-hand side of (26) in S1. Line 100 puts the right-hand side of (25) in VINC(I).

Since the coordinates (x_N, y_N) and (x_{N+1}, y_{N+1}) in [1] are given by

$$\left.\begin{aligned} (x_N, y_N) &= (0, -w) \\ (x_{N+1}, y_{N+1}) &= (0, w) \end{aligned}\right\} \tag{27}$$

it is evident from [1, eqs. (55) and (56)] that

$$\left.\begin{aligned} kx_N^+ &= 0 \\ ky_N^+ &= 0 \end{aligned}\right\} \tag{28}$$

It is also evident from [1, eq. (53)] that

$$\left.\begin{aligned} k(x_{N+1} - x_N) &= 0 \\ k(y_{N+1} - y_N) &= \gamma_N \end{aligned}\right\} \tag{29}$$

Taking advantage of (28) and (29), lines 107 to 110 put $4kV_N^{inc}$ of (25) in VINC(N).

It is evident from (27) and (29) that

$$kw = 0.5\gamma_N \tag{30}$$

Multiplying both sides of [1, eq. (74)] by 4k, replacing i by I therein, changing the variables of integration to make the ranges of integration from -1 to 1, approximating the integrals by means of (18), then substituting (30) for kw and [1, eqs. (57) and (58)] for the cosines and sines, we obtain

$$4kV_I^a = -2j\,\frac{\gamma_I\,\gamma_N}{4}\sum_{K=1}^{NG} A_K^{(NG)}(XX)\sum_{L=1}^{NG}\frac{A_L^{(NG)}\sqrt{1-(X_L^{(NG)})^2}\;H_1^{(2)}(\alpha_{INKL})}{\alpha_{INKL}} \tag{31}$$

where

$$\alpha_{INKL} = \sqrt{(XX)^2 + (YY)^2} \tag{32}$$

In (32), XX is the right-hand side of (21) with J replaced by N and YY is the right-hand side of (22) with J replaced by N. According to (28) and (29), the XX that appears explicitly in (31) is the same as the one in (32). When J=N, DO loop 39 puts the right-hand side of (31) in VA(I). When J=N, nested DO loops 40 and 41 perform the double summation in (31). The DO loop indices K and L obtain, respectively, the summation indices K and L in (31). DO loop 41 accumulates in V2 the summation with respect to L in (31). DO loop 40 accumulates in V1 the summation with respect to K in (31). Line 80 puts α_{INKL} of (32) in X1. Line 84 puts the Bessel functions $J_1(\alpha_{INKL})$ and $Y_1(\alpha_{INKL})$ in BJ(2) and BY(2), respectively. Line 85 uses (23) with n=1 and $x = \alpha_{INKL}$ to properly increment V2. Line 95 puts the right-hand side of (31) in VA(I). Line 105 obtains [1, eq. (68)] by setting VA(N) = 0.

In DO loop 21, $\frac{4k}{\eta} Z_{II}$ is put in Z((I-1)*N+I) where Z_{II} is given by [1, eq. (64)]. Thus, as the index I traverses its range, the diagonal elements of the square matrix $\frac{4k}{\eta}$ Z of [1, eq. (23)] are stored in the singly dimensioned array Z. Storage of $\frac{4k}{\eta}$ Z of [1, eq. (23)] is by columns in the singly dimensioned array Z. When I=N, C_A and C_ϕ of [1, eq. (86)] are put

in CA and CP, respectively, and $k\eta C^{inc}$ of [1, eq. (116)] is put in CINC.

The calculation of the diagonal elements of $\frac{4k}{\eta}$ Z of [1, eq. (23)] is described as follows. Replacing i by I in [1, eq. (64)], multiplying both sides of that equation by $4k/\eta$, changing the variables of integration therein to make the ranges of integration from -1 to 1, and then approximating both integrals by means of (18), we obtain

$$\frac{4k}{\eta} Z_{II} = \frac{\gamma_I^2}{4} \left\{ \left[4 + \frac{j4}{\pi}\left(3-2 \ln\left(0.5\gamma\, \gamma_I\right)\right)\right] + \sum_{K=1}^{NG} A_K^{(NG)} \sum_{L=1}^{NG} U1 \right\} \tag{33}$$

where

$$U1 = A_L^{(NG)} \left[H_o^{(2)}(XX) - 1 + \frac{j2}{\pi} \ln\left(0.5\gamma\, XX\right)\right] \tag{34}$$

where

$$XX = 0.5\gamma_I \left| X_K^{(NG)} - X_L^{(NG)} \right| \tag{35}$$

Prior to DO loop 21, line 115 puts γ in GAM. This γ is the e^{γ} that appears in [2, p. 2]. Line 122 puts in Z1 the square bracketed quantity in (33). Nested DO loops 22 and 23 perform the double summation in (33). The DO loop indices K and L obtain, respectively, the summation indices K and L in (33). DO loop 23 accumulates in Z2 the summation with respect to L in (33). Since the square bracketed quantity in [1, eq. (60)] approaches zero as x approaches zero, the term for which L=K in (33) is zero and therefore does not contribute to Z2. Line 136 puts XX of (35) in XX. Line 137 puts the Bessel functions $J_o(XX)$ and $Y_o(XX)$ in BJ(1) and BY(1), respectively. Line 138 uses (23) with n=0 and x=XX to put U1 of (34) in U1. Line 139 accumulates in Z2 the summation with respect to L in (33). Do loop 22 accumulates in Z1 the quantity in the { } braces in (33). Line 153 puts the right-hand side of (33) in Z((I-1)*N+I).

The calculation of C_A and C_ϕ of [1, eq. (86)] is described in this paragraph and the next paragraph. Substituting (30) into [1, eq. (89)] and using (18) to approximate both integrals in [1, eq. (89)], we obtain

$$C_A = I_A + \sum_{K=1}^{NG} A_K^{(NG)} \sqrt{1 - (X_K^{(NG)})^2} \sum_{L=1}^{NG} \sqrt{1 - (X_L^{(NG)})^2}\ U1 \tag{36}$$

where U1 is given by (34) in which XX is given by (35) with γ_I replaced by γ_N. Substitution of (30) into [1, eq. (96)] gives

$$I_A = \frac{\pi}{2} \left[\frac{\pi}{2} + j\left(\frac{1}{4} - \ln\left(\frac{\gamma\ \gamma_N}{8}\right)\right)\right] \tag{37}$$

Substituting (30) and [1, eq. (99)] into [1, eq. (90)] and using (18) to approximate both integrals in [1, eq. (90)], we obtain

$$C_\phi = j\pi + \sum_{K=1}^{NG} \frac{A_K^{(NG)}\ X_K^{(NG)}}{\sqrt{1 - (X_K^{(NG)})^2}} \sum_{L=1}^{NG} \frac{X_L^{(NG)}\ U1}{\sqrt{1 - (X_L^{(NG)})^2}} \tag{38}$$

where U1 is the same as in (36).

When I=N inside DO loop 21, C_A of (36) and C_ϕ of (38) are put in CA and CP, respectively. Line 124 puts I_A of (37) in CA. Line 125 puts $j\pi$ of (38) in CP. DO loop 23 accumulates in CA1 the sum with respect to L in (36). DO loop 23 also accumulates in CP1 the sum with respect to L in (38). Line 138 puts U1 of (36) in U1. Inside DO loop 22, lines 149 and 150 accumulate C_A of (36) and C_ϕ of (38) in CA and CP, respectively.

Multiplying [1, eq. (118)] by k', using (18) to approximate the integral therein, and substituting (30) for kw, we obtain

$$k' C^{inc} = 0.5\ \gamma_N \cos\phi^{inc} \sum_{K=1}^{NG} A_K^{(NG)} \sqrt{1 - (X_K^{(NG)})^2} \cos(0.5\ \gamma_N X_K^{(NG)} \sin\phi^{inc}) \tag{39}$$

When I=N inside DO loop 21, DO loop 22 accumulates in CINC the summation with respect to K in (39). Outside nested DO loops 21 and 22, line 156 performs the required multiplication by $0.5\Delta_N \cos\phi^{inc}$ so that $k\eta C^{inc}$ of (39) will finally be stored in CINC.

Line 162 puts in CURI(I) the Ith element of $\eta\vec{I}^{inc}$ where $\vec{I}^{inc}$ is the column vector that satisfies [1, eq. (23)]. Line 163 puts in CURA(I) the Ith element of $\eta\vec{I}^{a}$ where $\vec{I}^{a}$ is the column vector that satisfies [1, eq. (34)]. In the previous two sentences, I=1,2,...,N. Line 168 puts $2k\eta Y^{hs}$ of [1, eq. (86)] in YHS. Line 169 puts $8k\eta Y^{hs}$ in YAB. DO loop 26 adds to YAB the product of $4k\eta$ with the summation with respect to j in [1, eq. 115]. Thus, after exit from DO loop 26, $4k\eta(Y^{a} + Y^{b})$ of [1, eq. (115)] will reside in YAB. Line 173 sets VA(N) equal to ηY_N which, according to (30), is $2\pi kw$ so that $-8kC_J^-$ will reside in VA(J) for J=1,2,...,N. Here C_J^- is given by [1, eqs. (109) and (102)]. DO loop 27 accumulates in TI the product of $-8k\eta$ with the summation with respect to j in [1, eq. (116)]. Line 178 puts $4k\eta I$ of [1, eq. (116)] in TI. Line 179 puts V of [1, eq. (10)] in V.

Lines 183 to 188 perform the additional calculations that are necessary to determine V^{w} of [1, eq. (A-26)]. DO loop 35 sets VA(J) = 0 for J=1,2,...,N-1, so that, after exit from DO loop 35, $4k(V_J^{aw} - V_J^{bw})$ of [1, eq. (A-23)] will reside in VA(J) for J=1,2,...,N. Line 186 puts $\eta(I_N^{aw} - I_N^{bw})$ of [1, eq. (A-24)] in CURA(N). Line 187 puts $4k\eta(Y^{aw} + Y^{bw})$ of [1, eq. (A-25)] in YABW. Finally, line 188 puts V^{w} of [1, eq. (A-26)] in VW.

Our description of the main program for the methods of solution with and without pseudo-image is summarized in Table 2 where key variables in this program are listed. Each variable is identified by the line where it was read in, defined or incremented, and by its corresponding quantity in [1] or in the text.

Table 2. Key variables in the computer program, program lines where they are read in, defined or incremented, and their corresponding quantities in [1] or in the text.

Program variable	Program line	Corresponding quantity in [1] or in the text	Equation(s) where the quantity appears
N	11	N	[1, eq. (22)]
NG	11	NG	(18)
BK	11	k	[1, eq. (44)]
PINC	11	ϕ^{inc}	[1, eq. (44)]
X(I)	15	x_I	[1, eq. (53)]
Y(I)	19	y_I	[1, eq. (53)]
YO	22	POm and QOm	(7),(9),(12),(15),(14),(16)
XG(I)	25	$X_I^{(NG)}$	(18)
AG(I)	28	$A_I^{(NG)}$	(18)
CSP	32	$\cos \phi^{inc}$	[1, eq. (44)]
SNP	33	$\sin \phi^{inc}$	[1, eq. (44)]
DX(J)	44	$0.5k(x_{J+1} - x_J)$	(21)
DY(J)	45	$0.5k(y_{J+1} - y_J)$	(22)
G(J)	46	$0.5\gamma_J$	[1, eq. (53)]
XP(J)	47	kx_J^+	[1, eq. (55)]
YP(J)	48	ky_J^+	[1, eq. (56)]
X1	80	α_{IJKL}	(20)
BJ(1)	82	$J_o(\alpha_{IJKL})$	(19), (23)
BY(1)	82	$Y_o(\alpha_{IJKL})$	(19), (23)
Z2	86	The sum on L	(19)
Z1	88	The double sum	(19)
Z((J-1)*N+I)	93	$\frac{4k}{\eta} Z_{IJ}$	(19)
Z((I-1)*N+J)	102	$\frac{4k}{\eta} Z_{JI}$	(24), (19)
S1	96	S1	(26)
VINC(I)	100	$4kV_I^{inc}$	(25)
VINC(N)	110	$4kV_N^{inc}$	(25)
X1	80	α_{INKL}	(32)
BJ(2)	84	$J_1(\alpha_{INKL})$	(31), (23)
BY(2)	84	$Y_1(\alpha_{INKL})$	(31), (23)

Table 2. (continued)

Program variable	Program line	Corresponding quantity in [1] or in the text	Equation(s) where the quantity appears
V2	85	The sum on L	(31)
V1	89	The double sum	(31)
VA(I)	95	$4kV_I^a$	(31)
VA(N)	105	$4kV_N^a$	[1, eq. (68)]
GAM	115	γ	(33)
Z1	122	The square bracketed quantity	(33)
XX	136	XX	(35)
BJ(1)	137	$J_o(XX)$	(34), (23)
BY(1)	137	$Y_o(XX)$	(34), (23)
U1	138	U1	(34)
Z2	139	The sum on L	(33)
Z1	146	The quantity in the { } braces	(33)
Z((I-1)*N+I)	153	$\frac{4k}{\eta} Z_{II}$	(33)
CA	124	I_A	(37)
CP	125	$j\pi$	(38)
CA1	142	The sum on L	(36)
CP1	143	The sum on L	(38)
CA	149	C_A	(36)
CP	150	C_{π}	(38)
CINC	151	The sum on K	(39)
CINC	156	$k\eta C^{inc}$	(39)
CURI(I)	162	$\eta\ I_I^{inc}$	[1, eq. (23)]
CURA(I)	163	$\eta\ I_I^a$	[1, eq. (34)]

Table 2. (continued)

Program variable	Program line	Corresponding quantity in [1] or in the text	Equation(s) where the quantity appears
YHS	168	$2k\eta Y^{hs}$	[1, eq. (86)]
YAB	169	$8k\eta Y^{hs}$	[1, eq. (115)]
YAB	171	$4k\eta(Y^{a} + Y^{b})$	[1, eq. (115)]
VA(J)	95,173	$-8kC_J^-$	[1, eqs. (109) and (102)]
TI	176	$-8k\eta$ multiplied by the sum on j	[1, eq. (116)]
TI	178	$4k\eta I$	[1, eq. (116)]
V	179	V	[1, eq. (10)]
VA(J)	173,184	$4k(V_J^{aw} - V_J^{bw})$	[1, eq. (A-23)]
CURA(N)	186	$\eta(I_N^{aw} - I_N^{bw})$	[1, eq. (A-20)]
YABW	187	$4k\eta(Y^{aw} + Y^{bw})$	[1, eq. (A-25)]
VW	188	V^{w}	[1, eq. (A-26)]

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C     LISTING OF THE MAIN PROGRAM FOR THE METHODS OF
C     SOLUTION WITH AND WITHOUT PSEUDO-IMAGE
      COMPLEX U2,Z1,V1,Z2,V2,Z(1600),VA(40),VINC(40),CA
      COMPLEX CP,CA1,CP1,U1,CURI(40),CURA(40),YHS,YAB,TI
      COMPLEX V,YABW,VW
      COMMON Y0(33),PI2,PI4,PI7,BJ(100),BY(100)
      DIMENSION X(40),Y(40),XG(10),AG(10),DX(40)
      DIMENSION DY(40),G(40),XP(40),YP(40),IPS(40)
      OPEN(UNIT=20,FILE='MAUTZ1.DAT')
      OPEN(UNIT=21,FILE='MAUTZ2.DAT')
      READ(20,11) N,NG,BK,PINC
   11 FORMAT(2I3,2E14.7)
      WRITE(21,12) N,NG,BK,PINC
   12 FORMAT('   N NG',6X,'BK',11X,'PINC'/1X,2I3,2E14.7)
      READ(20,13) (X(I),I=1,N)
   13 FORMAT(5E14.7)
      WRITE(21,14) (X(I),I=1,N)
   14 FORMAT(' X'/(1X,5E14.7))
      READ(20,13) (Y(I),I=1,N)
      WRITE(21,15) (Y(I),I=1,N)
   15 FORMAT(' Y'/(1X,5E14.7))
      READ(20,13) (Y0(I),I=1,33)
      WRITE(21,16) (Y0(I),I=1,33)
   16 FORMAT(' Y0'/(1X,5E14.7))
      READ(20,13) (XG(I),I=1,NG)
      WRITE(21,17) (XG(I),I=1,NG)
   17 FORMAT(' XG'/(1X,5E14.7))
      READ(20,13) (AG(I),I=1,NG)
      WRITE(21,18) (AG(I),I=1,NG)
   18 FORMAT(' AG'/(1X,5E14.7))
      BK5=.5*BK
      CSP=COS(PINC)
      SNP=SIN(PINC)
      PI=3.141593
      PI2=2./PI
      PI4=PI/4.
      PI7=.75*PI
      U2=(0.,-2.)
      DO 37 J=1,N
      J1=J+1
      IF(J.EQ.N) J1=1
      D1=BK5*(X(J1)-X(J))
      D2=BK5*(Y(J1)-Y(J))
      DX(J)=D1
      DY(J)=D2
      G(J)=SQRT(D1*D1+D2*D2)
      XP(J)=BK5*(X(J1)+X(J))
      YP(J)=BK5*(Y(J1)+Y(J))
   37 CONTINUE
      NM=N-1
      DO 38 J=2,N
      II=(J-1)*N
      DXJ=DX(J)
```

```
      DYJ=DY(J)
      GJ=G(J)
      XPJ=XP(J)
      YPJ=YP(J)
      JM=J-1
      DO 39 I=1,JM
      DXI=DX(I)
      DYI=DY(I)
      GI=G(I)
      XPI=XP(I)
      YPI=YP(I)
      XIJ=XPI-XPJ
      YIJ=YPI-YPJ
      Z1=(0.,0.)
      IF(J.EQ.N) V1=(0.,0.)
      DO 40 K=1,NG
      AK=AG(K)
      XK=XIJ+XG(K)*DXI
      YK=YIJ+XG(K)*DYI
      Z2=(0.,0.)
      IF(J.EQ.N) V2=(0.,0.)
      DO 41 L=1,NG
      XL=XG(L)
      AL=AG(L)
      XX=XK-XL*DXJ
      YY=YK-XL*DYJ
      X1=SQRT(XX*XX+YY*YY)
      IF(J.EQ.N) GO TO 19
      CALL BES(0,X1)
      GO TO 20
19    CALL BES(1,X1)
      V2=AL*SQRT(1.-XL*XL)/X1*CMPLX(BJ(2),-BY(2))+V2
20    Z2=AL*CMPLX(BJ(1),-BY(1))+Z2
41    CONTINUE
      Z1=AK*Z2+Z1
      IF(J.EQ.N) V1=AK*XX*V2+V1
40    CONTINUE
      GG=GI*GJ
      II=II+1
      Z(II)=GG*Z1
      IF(J.NE.N) GO TO 42
      VA(I)=GG*U2*V1
      S1=DXI*CSP+DYI*SNP
      S3=1.
      IF(S1.NE.0.) S3=SIN(S1)/S1
      S2=XPI*CSP+YPI*SNP
      VINC(I)=8.*GI*S3*CMPLX(COS(S2),SIN(S2))
42    JJ=(I-1)*N+J
      Z(JJ)=Z(II)
39    CONTINUE
38    CONTINUE
      VA(N)=0.
      GN=G(N)
```

```
      S1=GN*SNP
      S3=1.
      IF(S1.NE.0.) S3=SIN(S1)/S1
      VINC(N)=8.*GN*S3
      WRITE(21,30)(VINC(I),I=1,N)
29    FORMAT(' VA'/(1X,4E14.7))
      WRITE(21,29)(VA(I),I=1,N)
30    FORMAT(' VINC'/(1X,4E14.7))
      GAM=1.781072
      GAM2=.5*GAM
      IZ=1
      NP=N+1
      DO 21 I=1,N
      GI=G(I)
      GAMG=GAM*GI
      Z1=CMPLX(4.,4./PI*(3.-2.*ALOG(GAMG)))
      IF(I.NE.N) GO TO 24
      CA=.5*PI*CMPLX(.5*PI,.25-ALOG(.25*GAMG))
      CP=CMPLX(0.,PI)
      CINC=0.
24    DO 22 K=1,NG
      Z2=(0.,0.)
      IF(I.NE.N) GO TO 25
      CA1=(0.,0.)
      CP1=(0.,0.)
25    XGK=XG(K)
      DO 23 L=1,NG
      IF(L.EQ.K) GO TO 23
      XGL=XG(L)
      XX=GI*ABS(XGK-XGL)
      CALL BES(0,XX)
      U1=AG(L)*CMPLX(BJ(1)-1.,-BY(1)+PI2*ALOG(GAM2*XX))
      Z2=U1+Z2
      IF(I.NE.N) GO TO 23
      SQ=SQRT(1.-XGL*XGL)
      CA1=SQ*U1+CA1
      CP1=XGL/SQ*U1+CP1
23    CONTINUE
      AGK=AG(K)
      Z1=AGK*Z2+Z1
      IF(I.NE.N) GO TO 22
      SQ=SQRT(1.-XGK*XGK)
      CA=AGK*SQ*CA1+CA
      CP=AGK*XGK/SQ*CP1+CP
      CINC=CINC+AGK*SQ*COS(S1*XGK)
22    CONTINUE
      Z(IZ)=GI*GI*Z1
      IZ=IZ+NP
21    CONTINUE
      CINC=GN*CSP*CINC
      WRITE(21,28)(Z(I),I=1,N)
28    FORMAT(' Z'/(1X,4E14.7))
      WRITE(21,34) CA,CP,CINC
```

```
34    FORMAT(' CA=',2E14.7,', CP=',2E14.7/' CINC=',E14.7)
      CALL DECOMP(N,IPS,Z)
      CALL SOLVE(N,IPS,Z,VINC,CURI)
      CALL SOLVE(N,IPS,Z,VA,CURA)
      WRITE(21,31) (CURI(I),I=1,N)
31    FORMAT(' CURI'/(1X,4E14.7))
      WRITE(21,32) (CURA(I),I=1,N)
32    FORMAT(' CURA'/(1X,4E14.7))
      YHS=GN*GN*CA-CP
      YAB=4.*YHS
      DO 26 J=1,NM
      YAB=YAB+CURA(J)*VA(J)
26    CONTINUE
      VA(N)=2.*PI*GN
      TI=(0.,0.)
      DO 27 J=1,N
      TI=TI+CURI(J)*VA(J)
27    CONTINUE
      TI=4.*CINC-.5*TI
      V=TI/YAB
      WRITE(21,33) YHS,YAB,TI,V
33    FORMAT(' YHS=',2E14.7,', YAB=',2E14.7/
     1 ' TI=',2E14.7,', V=',2E14.7)
      DO 35 J=1,NM
      VA(J)=0.
35    CONTINUE
      CALL SOLVE(N,IPS,Z,VA,CURA)
      YABW=.5*YAB+PI*GN*CURA(N)
      VW=TI/YABW
      WRITE(21,36) YABW,VW
36    FORMAT(' YABW=',2E14.7,', VW=',2E14.7)
      STOP
      END
```

```
C       LISTING OF THE INPUT DATA FILE MAUTZ1.DAT
C
 36 10 0.1570796E+01 0.3141593E+01
 0.0000000E+00 0.3026887E-01 0.8988691E-01 0.1770427E+00 0.2890879E+00
 0.4226183E+00 0.5735764E+00 0.7373757E+00 0.9090390E+00 0.1083350E+01
 0.1255014E+01 0.1418813E+01 0.1569771E+01 0.1703301E+01 0.1815347E+01
 0.1902502E+01 0.1962121E+01 0.1992389E+01 0.1992389E+01 0.1962121E+01
 0.1902502E+01 0.1815347E+01 0.1703301E+01 0.1569771E+01 0.1418813E+01
 0.1255014E+01 0.1083350E+01 0.9090390E+00 0.7373757E+00 0.5735764E+00
 0.4226183E+00 0.2890879E+00 0.1770427E+00 0.8988691E-01 0.3026887E-01
 0.0000000E+00
 0.8715574E-01 0.2588190E+00 0.4226183E+00 0.5735764E+00 0.7071068E+00
 0.8191520E+00 0.9063078E+00 0.9659258E+00 0.9961947E+00 0.9961947E+00
 0.9659258E+00 0.9063078E+00 0.8191520E+00 0.7071068E+00 0.5735764E+00
 0.4226183E+00 0.2588190E+00 0.8715574E-01-0.8715574E-01-0.2588190E+00
-0.4226183E+00-0.5735764E+00-0.7071068E+00-0.8191520E+00-0.9063078E+00
-0.9659258E+00-0.9961947E+00-0.9961947E+00-0.9659258E+00-0.9063078E+00
-0.8191520E+00-0.7071068E+00-0.5735764E+00-0.4226183E+00-0.2588190E+00
-0.8715574E-01
-0.3072582E+04 0.7368758E+04-0.6085100E+03 0.1710234E+02-0.2271001E+00
 0.1600171E-02-0.5961089E-05 0.9545773E-08 0.4163150E+05 0.3420211E+03
 0.1000000E+01-0.6024727E+04 0.1613512E+04-0.7532210E+02 0.1402590E+01
-0.1275602E-01 0.5832787E-04-0.1107698E-06 0.3072946E+05 0.2886431E+03
 0.1000000E+01 0.9999999E+00-0.1097659E-02 0.2461455E-04 0.1000000E+01
 0.1829893E-02-0.3191328E-04-0.1562498E-01 0.1427079E-03-0.5937434E-05
 0.4687498E-01-0.1998720E-03 0.7317495E-05
-0.9739065E+00-0.8650634E+00-0.6794096E+00-0.4333954E+00-0.1488743E+00
 0.1488743E+00 0.4333954E+00 0.6794096E+00 0.8650634E+00 0.9739065E+00
 0.6667134E-01 0.1494513E+00 0.2190864E+00 0.2692667E+00 0.2955242E+00
 0.2955242E+00 0.2692667E+00 0.2190864E+00 0.1494513E+00 0.6667134E-01
```

```
C        LISTING OF THE OUTPUT DATA FILE MAUTZ2.DAT
C
  N NG        BK              PINC
 36 10 0.1570796E+01 0.3141593E+01
X
 0.0000000E+00 0.3026887E-01 0.8988691E-01 0.1770427E+00 0.2890879E+00
 0.4226183E+00 0.5735764E+00 0.7373757E+00 0.9090390E+00 0.1083350E+01
 0.1255014E+01 0.1418813E+01 0.1569771E+01 0.1703301E+01 0.1815347E+01
 0.1902502E+01 0.1962121E+01 0.1992389E+01 0.1992389E+01 0.1962121E+01
 0.1902502E+01 0.1815347E+01 0.1703301E+01 0.1569771E+01 0.1418813E+01
 0.1255014E+01 0.1083350E+01 0.9090390E+00 0.7373757E+00 0.5735764E+00
 0.4226183E+00 0.2890879E+00 0.1770427E+00 0.8988691E-01 0.3026887E-01
 0.0000000E+00
Y
 0.8715574E-01 0.2588190E+00 0.4226183E+00 0.5735764E+00 0.7071068E+00
 0.8191520E+00 0.9063078E+00 0.9659258E+00 0.9961947E+00 0.9961947E+00
 0.9659258E+00 0.9063078E+00 0.8191520E+00 0.7071068E+00 0.5735764E+00
 0.4226183E+00 0.2588190E+00 0.8715574E-01-0.8715574E-01-0.2588190E+00
-0.4226183E+00-0.5735764E+00-0.7071068E+00-0.8191520E+00-0.9063078E+00
-0.9659258E+00-0.9961947E+00-0.9961947E+00-0.9659258E+00-0.9063078E+00
-0.8191520E+00-0.7071068E+00-0.5735764E+00-0.4226183E+00-0.2588190E+00
-0.8715574E-01
Y0
-0.3072582E+04 0.7368758E+04-0.6085100E+03 0.1710234E+02-0.2271001E+00
 0.1600171E-02-0.5961089E-05 0.9545773E-08 0.4163150E+05 0.3420211E+03
 0.1000000E+01-0.6024727E+04 0.1613512E+04-0.7532210E+02 0.1402590E+01
-0.1275602E-01 0.5832787E-04-0.1107698E-06 0.3072946E+05 0.2886431E+03
 0.1000000E+01 0.9999999E+00-0.1097659E-02 0.2461455E-04 0.1000000E+01
 0.1829893E-02-0.3191328E-04-0.1562498E-01 0.1427079E-03-0.5937434E-05
 0.4687498E-01-0.1998720E-03 0.7317495E-05
XG
-0.9739065E+00-0.8650634E+00-0.6794096E+00-0.4333954E+00-0.1488743E+00
 0.1488743E+00 0.4333954E+00 0.6794096E+00 0.8650634E+00 0.9739065E+00
AG
 0.6667134E-01 0.1494513E+00 0.2190864E+00 0.2692667E+00 0.2955242E+00
 0.2955242E+00 0.2692667E+00 0.2190864E+00 0.1494513E+00 0.6667134E-01
VINC
 0.1094818E+01-0.2603224E-01 0.1089960E+01-0.1031663E+00
 0.1070414E+01-0.2277548E+00 0.1021332E+01-0.3915596E+00
 0.9268360E+00-0.5797543E+00 0.7749375E+00-0.7703197E+00
 0.5626479E+00-0.9361391E+00 0.2993480E+00-0.1050082E+01
 0.6526236E-02-0.1091791E+01-0.2867749E+00-0.1053589E+01
-0.5514164E+00-0.9427966E+00-0.7656732E+00-0.7795281E+00
-0.9198373E+00-0.5907919E+00-0.1016581E+01-0.4037424E+00
-0.1067612E+01-0.2405345E+00-0.1088651E+01-0.1161893E+00
-0.1094428E+01-0.3911872E-01-0.1095153E+01-0.1309429E-01
-0.1094428E+01-0.3911890E-01-0.1088651E+01-0.1161897E+00
-0.1067612E+01-0.2405351E+00-0.1016581E+01-0.4037431E+00
-0.9198368E+00-0.5907926E+00-0.7656725E+00-0.7795288E+00
-0.5514155E+00-0.9427971E+00-0.2867739E+00-0.1053589E+01
 0.6527342E-02-0.1091791E+01 0.2993491E+00-0.1050081E+01
 0.5626488E+00-0.9361386E+00 0.7749382E+00-0.7703190E+00
 0.9268364E+00-0.5797535E+00 0.1021332E+01-0.3915590E+00
```

```
 0.1070414E+01-0.2277543E+00 0.1089960E+01-0.1031659E+00
 0.1094818E+01-0.2603204E-01 0.1095231E+01 0.0000000E+00
VA
 0.2994785E-01-0.1381276E-02 0.2858944E-01-0.5333051E-02
 0.2962683E-01-0.1131610E-01 0.2911258E-01-0.1854561E-01
 0.2649176E-01-0.2613214E-01 0.2176669E-01-0.3322932E-01
 0.1528127E-01-0.3915687E-01 0.7584020E-02-0.4347779E-01
-0.6980846E-03-0.4602249E-01-0.8966518E-02-0.4686616E-01
-0.1672587E-01-0.4627063E-01-0.2362260E-01-0.4461700E-01
-0.2944913E-01-0.4233324E-01-0.3412371E-01-0.3983858E-01
-0.3765389E-01-0.3750189E-01-0.4009859E-01-0.3562038E-01
-0.4152818E-01-0.3440646E-01-0.4199799E-01-0.3398786E-01
-0.4152818E-01-0.3440646E-01-0.4009859E-01-0.3562038E-01
-0.3765389E-01-0.3750189E-01-0.3412371E-01-0.3983858E-01
-0.2944913E-01-0.4233324E-01-0.2362260E-01-0.4461700E-01
-0.1672587E-01-0.4627063E-01-0.8966518E-02-0.4686616E-01
-0.6980846E-03-0.4602249E-01 0.7584020E-02-0.4347779E-01
 0.1528127E-01-0.3915687E-01 0.2176669E-01-0.3322932E-01
 0.2649176E-01-0.2613214E-01 0.2911258E-01-0.1854561E-01
 0.2962683E-01-0.1131610E-01 0.2858944E-01-0.5333051E-02
 0.2994785E-01-0.1381276E-02 0.0000000E+00 0.0000000E+00
Z
 0.7473691E-01 0.1385026E+00 0.7335540E-01 0.7065693E-01
 0.6932754E-01 0.2968639E-01 0.6298721E-01 0.6236447E-02
 0.5483766E-01-0.1027953E-01 0.4548406E-01-0.2216265E-01
 0.3555794E-01-0.3037457E-01 0.2564739E-01-0.3554059E-01
 0.1624363E-01-0.3820945E-01 0.7709407E-02-0.3891708E-01
 0.2694717E-03-0.3818308E-01-0.5978546E-02-0.3649254E-01
-0.1104134E-01-0.3427249E-01-0.1499958E-01-0.3187981E-01
-0.1797623E-01-0.2959581E-01-0.2010926E-01-0.2763171E-01
-0.2152729E-01-0.2613496E-01-0.2233387E-01-0.2520174E-01
-0.2259513E-01-0.2488496E-01-0.2233394E-01-0.2520178E-01
-0.2152726E-01-0.2613493E-01-0.2010927E-01-0.2763173E-01
-0.1797624E-01-0.2959584E-01-0.1499953E-01-0.3187969E-01
-0.1104136E-01-0.3427249E-01-0.5978571E-02-0.3649271E-01
 0.2694886E-03-0.3818303E-01 0.7709373E-02-0.3891693E-01
 0.1624367E-01-0.3820957E-01 0.2564739E-01-0.3554058E-01
 0.3555793E-01-0.3037456E-01 0.4548407E-01-0.2216266E-01
 0.5483765E-01-0.1027953E-01 0.6298724E-01 0.6236455E-02
 0.6932755E-01 0.2968641E-01 0.7335536E-01 0.7065692E-01
CA= 0.2461616E+01 0.4775566E+01, CP= 0.1841846E-01 0.3171688E+01
CINC=-0.2151464E+00
CURI
 0.2113638E+01-0.5360954E+00 0.1998972E+01-0.6872047E+00
 0.1791495E+01-0.9038445E+00 0.1479588E+01-0.1134857E+01
 0.1068554E+01-0.1316476E+01 0.5917228E+00-0.1386905E+01
 0.1110202E+00-0.1306445E+01-0.2971143E+00-0.1075142E+01
-0.5667900E+00-0.7376537E+00-0.6671781E+00-0.3697278E+00
-0.6132755E+00-0.5098838E-01-0.4569358E+00 0.1628754E+00
-0.2636858E+00 0.2552014E+00-0.8820872E-01 0.2457596E+00
 0.3996931E-01 0.1756236E+00 0.1167466E+00 0.8950505E-01
 0.1534665E+00 0.2329606E-01 0.1637747E+00-0.1162453E-02
 0.1534665E+00 0.2329659E-01 0.1167467E+00 0.8950505E-01
```

```
 0.3996842E-01 0.1756239E+00-0.8820911E-01 0.2457597E+00
-0.2636863E+00 0.2552012E+00-0.4569371E+00 0.1628750E+00
-0.6132758E+00-0.5098913E-01-0.6671781E+00-0.3697297E+00
-0.5667889E+00-0.7376546E+00-0.2971131E+00-0.1075144E+01
 0.1110217E+00-0.1306445E+01 0.5917245E+00-0.1386905E+01
 0.1068555E+01-0.1316475E+01 0.1479589E+01-0.1134857E+01
 0.1791496E+01-0.9038439E+00 0.1998973E+01-0.6872036E+00
 0.2113638E+01-0.5360951E+00 0.2149824E+01-0.4822035E+00
CUBA
 0.8435730E-01-0.2000979E+00 0.7689659E-01-0.7414376E-02
 0.6570814E-01-0.4870010E-01 0.5236392E-01-0.5136077E-01
 0.3856174E-01-0.5534811E-01 0.2580129E-01-0.5480544E-01
 0.1514947E-01-0.5149234E-01 0.7139070E-02-0.4639276E-01
 0.1799082E-02-0.4062632E-01-0.1219058E-02-0.3507616E-01
-0.2472905E-02-0.3032159E-01-0.2568123E-02-0.2663050E-01
-0.2039613E-02-0.2401857E-01-0.1288466E-02-0.2233653E-01
-0.5694495E-03-0.2135902E-01-0.1499719E-04-0.2085441E-01
 0.3240804E-03-0.2063298E-01 0.4371095E-03-0.2057321E-01
 0.3240795E-03-0.2063297E-01-0.1499475E-04-0.2085441E-01
-0.5694595E-03-0.2135902E-01-0.1288466E-02-0.2233653E-01
-0.2039618E-02-0.2401856E-01-0.2568109E-02-0.2663050E-01
-0.2472912E-02-0.3032158E-01-0.1219056E-02-0.3507617E-01
 0.1799082E-02-0.4062631E-01 0.7139078E-02-0.4639276E-01
 0.1514947E-01-0.5149233E-01 0.2580127E-01-0.5480543E-01
 0.3856175E-01-0.5534811E-01 0.5236392E-01-0.5136079E-01
 0.6570815E-01-0.4870008E-01 0.7689659E-01-0.7414370E-02
 0.8435730E-01-0.2000979E+00 0.8697801E-01 0.3527458E+00
YHS= 0.2771880E-01-0.3082181E+01, YAB= 0.9196463E-01-0.1235218E+02
TI=-0.1835366E+01 0.4155412E+00, V=-0.3474545E-01-0.1483277E+00
YABW= 0.8403454E-01-0.1077051E+02, VW=-0.3990850E-01-0.1700951E+00
```

V. THE SUBROUTINE BESJ1

The subroutine BESJ1(N, X, BJ1) puts J_1(I*X) in BJ1(I) for I=1,2,...,N. Here, J_1 is the Bessel function of the first kind of order one, N is an integer not less than one, and X is a non-negative real number. The minimum allocation for BJ1 is given by

DIMENSION BJ1(N)

The subroutine BESJ1 is listed at the end of this section. If I*X$\leq$3, then BJ1(I) is calculated by lines 7 to 20 inside DO loop 15 of the listing. As suggested in [2, Sec. 9.12., Example 1], J_{MZ}(XX) and J_{MZ-1}(XX) are set equal to the arbitrary values of zero and 10^{-20}, respectively, where

$$XX = I*X \tag{40}$$

and

$$MZ = 11 + 2[XX] \tag{41}$$

where [XX] is the largest integer that does not exceed XX. This is done in lines 7 to 9. In DO loop 16, the recurrence relation (4) is used to calculate $\{J_n(XX),\ n = MZ-2, MZ-3,\ldots,0\}$. The calculated value of J_1(XX) has to be divided by α of (5). This division is done in line 20.

If I*X > 3, then BJ1(I) is calculated by lines 22 to 32 inside DO loop 15. The polynomial approximation [2, eq. (9.4.6)] is used. With regard to [2, eq. (9.4.6)], lines 28 and 29 put f_1 in F, lines 30 and 31 put θ_1 in T, and line 32 puts J_1(I*X) in BJ1(I).

```
C      LISTING OF THE SUBROUTINE BESJ1
       SUBROUTINE BESJ1(N,X,BJ1)
       DIMENSION BJ(18),BJ1(10000)
       DO 15 I=1,N
       XX=FLOAT(I)*X
       IF(XX-3.) 17,17,18
17     MZ=11+2*IFIX(XX)
       BJ(MZ+1)=0.
       BJ(MZ)=1.E-20
       M1=MZ-1
       X2=2./XX
       DO 16 K=1,M1
       MK=MZ-K
       BJ(MK)=X2*FLOAT(MK)*BJ(MK+1)-BJ(MK+2)
16     CONTINUE
       ALP=.5*BJ(1)
       DO 19 J=3,MZ,2
       ALP=ALP+BJ(J)
19     CONTINUE
       BJ1(I)=BJ(2)/(2.*ALP)
       GO TO 15
18     X1=3./XX
       X2=X1*X1
       X3=X2*X1
       X4=X3*X1
       X5=X4*X1
       X6=X5*X1
       F=.7978846+.156E-05*X1+.1659667E-01*X2+.17105E-03*X3-
     1 .249511E-02*X4+.113653E-02*X5-.20033E-03*X6
       T=XX-2.356194+.1249961*X1+.565E-04*X2-.637879E-02*X3+
     1 .74348E-03*X4+.79824E-03*X5-.29166E-03*X6
       BJ1(I)=F/SQRT(XX)*COS(T)
15     CONTINUE
       RETURN
       END
```

VI. THE SUBROUTINE BESJY

The subroutine BESJY(N1, N2, X, BJY) puts $J_N(X)\ Y_N(X)$ in BJY(N) for N=N1, N1+1, ..., N2 where J_N and Y_N are, respectively, the Bessel functions of the first and second kind of order N. Moreover, X is a non-negative real number, N2 $\geq$ N1, and N1 is an integer appreciably larger than X so that Debye's asymptotic expansions [2, eqs. (9.3.7) and (9.3.8)] apply to $J_{N1}(X)$ and $Y_{N1}(X)$. The minimum allocation for BJY is given by

```
DIMENSION BJY(N2)
```

Combining [2, eqs. (9.3.7) to (9.3.9)], we obtain

$$J_N(X)Y_N(X) = -\frac{t}{\pi N}\left(1 + \sum_{i=1}^{4} \frac{u_i(t)}{N^i}\right)\left(1 + \sum_{i=1}^{4} \frac{(-1)^i u_i(t)}{N^i}\right) \tag{42}$$

where

$$t = \frac{1}{\sqrt{1 - (X/N)^2}} \tag{43}$$

and

$$u_1(t) = (3t - 5t^3)/24 \tag{44}$$

$$u_2(t) = (81t^2 - 462t^4 + 385t^6)/1152 \tag{45}$$

$$u_3(t) = (30375t^3 - 369603t^5 + 765765t^7 - 425425t^9)/414720 \tag{46}$$

$$u_4(t) = (4465125t^4 - 94121676t^6 + 349922430t^8 - 446185740t^{10} + 185910725t^{12})/39813120 \tag{47}$$

Equation (43) is verified in the following manner. From [2, eq. (9.3.7)],

$$t = \coth \alpha \tag{48}$$

Substituting

$$\operatorname{sech} \alpha = X/N \tag{49}$$

into the identity [2, eq. (4.5.17)]

$$\tanh \alpha = \sqrt{1 - \operatorname{sech}^2 \alpha} \tag{50}$$

and noting that coth α is the reciprocal of tanh α, we obtain

$$\coth \alpha = \frac{1}{\sqrt{1 - (X/N)^2}} \tag{51}$$

Substitution of (51) into (48) gives the desired result (43).

The subroutine BESJY is listed at the end of this section. The right-hand side of (42) is calculated inside DO loop 11 of this listing. The order N that appears in (42) is obtained as the index of DO loop 11. Lines 5 to 8 put N, N^2, N^3, and N^4 in F1, F2, F3, and F4, respectively. With regard to (43)-(47), lines 10 to 20 put t^I in TI for I=1,2,3,4,5,6,7,8,9,10,12. Line 21 puts $u_1(t)/N$ in U1, line 22 puts $u_2(t)/N^2$ in U2, line 23 puts $u_3(t)/N^3$ in U3, and lines 24 and 25 put $u_4(t)/N^4$ in U4. Line 28 puts the right-hand side of (42) in BJY(N).

```
C         LISTING OF THE SUBROUTINE BESJY
          SUBROUTINE BESJY(N1,N2,X,BJY)
          DIMENSION BJY(10000)
          DO 11 N=N1,N2
          F1=N
          F2=F1*F1
          F3=F2*F1
          F4=F3*F1
          XN=X/F1
          T1=1./SQRT(1.-XN*XN)
          T2=T1*T1
          T3=T2*T1
          T4=T3*T1
          T5=T4*T1
          T6=T5*T1
          T7=T6*T1
          T8=T7*T1
          T9=T8*T1
          T10=T9*T1
          T12=T10*T2
          U1=(3.*T1-5.*T3)/(24.*F1)
          U2=(81.*T2-462.*T4+385.*T6)/(1152.*F2)
          U3=(30375.*T3-369603.*T5+765765.*T7-425425.*T9)/(414720.*F3)
          U4=(4465125.*T4-9412168.E+01*T6+3499224.E+02*T8-4461857.E+02*
        1 T10+1859107.E+02*T12)/(3981312.E+01*F4)
          U5=1.+U2+U4
          U6=U1+U3
          BJY(N)=-T1/(3.141593*F1)*(U5+U6)*(U5-U6)
     11   CONTINUE
          RETURN
          END
```

VII. THE MAIN PROGRAM FOR THE FOURIER SERIES METHOD OF SOLUTION

The main program for the Fourier series method of solution calculates the complex constant V of [1, eq. (B-25)]. Input data are read from the file MAUTZ3.DAT, output data are written on the file MAUTZ4.DAT, and the subroutines BES, BESJ1, and BESJY are called. These subroutines are described in Sections II, V, and VI.

The main program for the Fourier series method of solution is listed at the end of this section. In the open statements on lines 6 and 7 of this listing, MAUTZ3.DAT is the input data file and MAUTZ4.DAT is the output data file. The input data are read early in the program according to

```
      READ(20,17) N1, N2, X, P
17    FORMAT(I3, I5, 2E14.7)
      READ(20,11)(YO(I), I = 1, 33)
11    FORMAT(5E14.7)
```

The definitions of N1 and N2 are based on the technical digression in the next paragraph.

Truncating the infinite series in [1, eq. (B-25)], we obtain

$$V = -\frac{2\ U1}{\phi_o U2}\, e^{-jka\cos\phi_o} \tag{52}$$

where

$$U1 = \frac{1}{4H_o^{(2)}(ka)} + \sum_{N=1}^{N1-1} \frac{j^N}{H_N^{(2)}(ka)} \left(\frac{J_1(N\phi_o)}{N\phi_o}\right) \tag{53}$$

and

$$U2 = \frac{1}{8J_o(ka)H_o^{(2)}(ka)} + \sum_{N=1}^{N1-1} \frac{1}{J_N(ka)H_N^{(2)}(ka)} \left(\frac{J_1(N\phi_o)}{N\phi_o}\right)^2$$

$$+\ j \sum_{N=N1}^{N2} \frac{1}{J_N(ka)Y_N(ka)} \left(\frac{J_1(N\phi_o)}{N\phi_o}\right)^2 \tag{54}$$

In (53) and (54), $H_N^{(2)}$ is the Hankel function of the second kind of order N. According to (23),

$$H_N^{(2)}(ka) = J_N(ka) - jY_N(ka) \tag{55}$$

where J_N and Y_N are, respectively, the Bessel functions of the first and second kinds of order N. Still in (53) and (54), N1 and N2 are positive integers such that

$$N2 \geq N1 \tag{56}$$

$$N1 >> ka \tag{57}$$

$$N2 >> \pi/\phi_o \tag{58}$$

The inequality (57) allows the series in (53) to be truncated at N=N1-1. It allows deletion of the quantity $J_N(ka)$ on the right-hand side of (55) whenever $N \geq N1$, so that $j/Y_N(ka)$ instead of $1/H_N^{(2)}(ka)$ could be put in the second summation on the right-hand side of (54). The inequality (58) is necessary because, in order for [1, eq. (B-9)] to be well-satisfied, $e^{jm\phi}$ terms whose periods are appreciably smaller than $2\phi_o$ must be retained on the left-hand side of [1, eq. (B-9)]. The period of $e^{jm\phi}$ is $2\phi_o$ when m is π/ϕ_o.

The N1 and N2 that appear in the first read statement are the same as those in (53) and (54). In the same read statement, X is the electrical length ka in (53) and (54), and P is the angle ϕ_o in (53) and (54). Here, P is in radians. In the second read statement, the array YO contains input data for the subroutine BES. The values of the elements of YO were given in Section II. These values should not deeply concern the user because he will never have to change them.

Minimum allocations are given by

```
COMMON BJ(N1), BY(N1)
DIMENSION BJ1(N2), BJY(N2)
```

Immediately after the main program at the end of this section, the contents of the input data file MAUTZ3.DAT and the output data file MAUTZ4.DAT are listed when

$$\left.\begin{aligned} N1 &= 20 \\ N2 &= 10000 \\ X &= \pi/2 \\ P &= \pi/36 \end{aligned}\right\} \tag{59}$$

The values of X and P in (59) obtain the example of [1, Section VIII]. These values mean that

$$ka = \pi/2 \tag{60}$$

and

$$\pi/\phi_o = 36 \tag{61}$$

In view of (60) and (61), the values of N1 and N2 in (59) satisfy the inequalities (57) and (58). The output data file MAUTZ4.DAT contains all the data put out by the write statements in the main program for the Fourier series method of solution. The contents of the output data file MAUTZ4.DAT are described in the next three paragraphs.

The input data are written out immediately after they are read in. The jth number in the ith row under the heading "BJ" is $J_{5(i-1)+j-1}(ka)$. The jth number in the ith row under the heading "BY" is $Y_{5(i-1)+j-1}(ka)$. By means of (55), the Bessel functions mentioned in the previous two sentences are used to construct the $H_o^{(2)}(ka)$ and the $H_N^{(2)}(ka)$ that appear

in (53). The number preceded by "BJ1(1)=" is $J_1(\phi_o)$, and the number preceded by "BJ1(N2)=" is $J_1(N2*\phi_o)$.

Each line under the heading "N U1 U2" contains five numbers. The second and third numbers are the real and imaginary parts of the right-hand side of (53) with N1-1 replaced by the first number. The fourth and fifth numbers are the real and imaginary parts of the right-hand side of (54) with the maximum value of N therein equal to the first number.

The number preceded by "BJY(N1)=" is $J_{N1}(ka)Y_{N1}(ka)$, and the number preceded by "BJY(N2)=" is $J_{N2}(ka)Y_{N2}(ka)$. The numbers written under the heading "BJ1" are

$$\sum_{N=N1}^{I} \frac{1}{J_N(ka)\ Y_N(ka)} \left(\frac{J_1(N\phi_o)}{N\phi_o}\right)^2 \tag{62}$$

where I runs from N1 to N2 in steps of 50. These numbers are written in order to observe the rate at which (62) converges as I increases. Finally, the two numbers preceded by "V=" are, respectively, the real and imaginary parts of V of (52).

Lines 17 to 19 define the common variables (2) that are input data for the subroutine BES. Line 21 puts $J_N(ka)$ in BJ(N+1) for N=0,1,...,N1-1. Line 21 also puts $Y_N(ka)$ in BY(N+1) for N=0,1,...,N1-1. Line 26 puts $J_1(N\phi_o)$ in BJ1(N) for N=1,2,...,N2.

Line 30 puts in U1 the term not governed by the summation sign in (53). Line 31 puts in U2 the term not governed by either one of the summation signs in (54). Inside DO loop 13, line 47 adds to U1 the term whose summation index is N in (53), and line 48 adds to U2 the term whose summation index is N in (54). Line 39 puts $\left(\frac{J_1(N\phi_o)}{N\phi_o}\right)$ in B. Line 41 puts $J_N(ka)$ in

BJN. Lines 42 to 45 put $(\frac{J_1(N\phi_o)}{N\phi_o})/H_N^{(2)}(ka)$ in U3. If $|J_N(ka)| \leq 10^{-10}$, then, as a measure to avoid an underflow, $H_N^{(2)}(ka)$ is approximated by $-jY_N(ka)$. After line 46 is executed, j^N will reside in U.

Line 51 puts $J_N(ka)\ Y_N(ka)$ in BJY(N) for N=N1, N1+1,..., N2. Inside DO loop 14, line 58 accumulates in S the second summation on the right-hand side of (54). The index N of DO loop 14 obtains the summation index N of (54). Line 57 puts $(\frac{J_1(N\phi_o)}{N\phi_o})$ in B. Line 63 puts the right-hand side of (54) in U2. Finally, line 65 puts V of (52) in V.

Our description of the main program for the Fourier series method of solution is summarized in Table 3 where key variables in this program are listed. Each variable is identified by the line where it was read in, defined, or incremented, and by its corresponding quantity in the text.

Table 3. Key variables in the computer program, program lines where they are read in, defined, or incremented, and their corresponding quantities in the text.

Program variables	Program line	Corresponding quantity in the text	Equation(s) where the quantity appears
N1	8	N1	(53)-(54)
N2	8	N2	(54)
X	8	ka	(52)-(55)
P	8	ϕ_o	(52)-(54)
YO	12	POm and QOm	(7),(9),(12),(15),(14),(16)
BJ(N+1)	21	$J_N(ka)$	(53)-(55)
BY(N+1)	21	$Y_N(ka)$	(53)-(55)
BJ1(N)	26	$J_1(N\phi_o)$	(53)-(54)
U1	30	$1/(4H_o^{(2)}(ka))$	(53)
U2	31	$1/(8J_o(ka)H_o^{(2)}(ka))$	(54)
B	39	$\dfrac{J_1(N\phi_o)}{N\phi_o}$	(53)-(54)
BJN	41	$J_N(ka)$	(53)-(55)
U3	42-45	$(\dfrac{J_1(N\phi_o)}{N\phi_o})/H_N^{(2)}(ka)$	(53)-(54)
U	46	j^N	(53)
U1	47	U1	(53)
U2	48	U2 without $j \sum_{N=N1}^{N2}$	(54)
BJY(N)	51	$J_N(ka)Y_N(ka)$	(54)
B	57	$J_1(N\phi_o)/N\phi_o$	(54)
S	58	$\sum_{N=N1}^{N2}$	(54)
U2	63	U2	(54)
V	65	V	(52)

```
C         LISTING OF THE MAIN PROGRAM FOR THE
C         FOURIER SERIES METHOD OF SOLUTION
          COMPLEX U1,U2,U,U3,V
          COMMON Y0(33),PI2,PI4,PI7,BJ(100),BY(100)
          DIMENSION BJ1(10000),BJY(10000)
          OPEN(UNIT=20,FILE='MAUTZ3.DAT')
          OPEN(UNIT=21,FILE='MAUTZ4.DAT')
          READ(20,17) N1,N2,X,P
17        FORMAT(I3,I5,2E14.7)
          WRITE(21,27) N1,N2,X,P
27        FORMAT(' N1=',I3,', N2=',I5,', X=',E14.7,', P=',E14.7)
          READ(20,11) (Y0(I),I=1,33)
11        FORMAT(5E14.7)
          WRITE(21,12) (Y0(I),I=1,33)
12        FORMAT(' Y0'/(1X,5E14.7))
          PI=3.141593
          PI2=2./PI
          PI4=PI/4.
          PI7=.75*PI
          N1M=N1-1
          CALL BES(N1M,X)
          WRITE(21,18) (BJ(I),I=1,N1)
18        FORMAT(' BJ'/(1X,5E14.7))
          WRITE(21,19) (BY(I),I=1,N1)
19        FORMAT(' BY'/(1X,5E14.7))
          CALL BESJ1(N2,P,BJ1)
          WRITE(21,20) BJ1(1),BJ1(N2)
20        FORMAT(' BJ1(1)=',E14.7,', BJ1(N2)=',E14.7)
          N=0
          U1=.25/CMPLX(BJ(1),-BY(1))
          U2=.5/BJ(1)*U1
          WRITE(21,15)
15        FORMAT('   N',14X,'U1',26X,'U2')
          WRITE(21,25) N,U1,U2
25        FORMAT(1X,I3,4E14.7)
          U=(1.,0.)
          DO 13 N=1,N1M
          FN=N*P
          B=BJ1(N)/FN
          NP=N+1
          BJN=BJ(NP)
          IF(ABS(BJN)-1.E-10) 22,22,23
22        U3=B/BY(NP)*(0.,1.)
          GO TO 24
23        U3=B/CMPLX(BJN,-BY(NP))
24        U=U*(0.,1.)
          U1=U*U3+U1
          U2=B/BJN*U3+U2
          WRITE(21,25) N,U1,U2
13        CONTINUE
          CALL BESJY(N1,N2,X,BJY)
          WRITE(21,21) BJY(N1),BJY(N2)
21        FORMAT(' BJY(N1)=',E14.7,', BJY(N2)=',E14.7)
```

```
      S=0.
      DO 14 N=N1,N2
      FN=N*P
      B=BJ1(N)/FN
      S=S+B*B/BJY(N)
      BJ1(N)=S
   14 CONTINUE
      WRITE(21,16) (BJ1(I),I=N1,N2,50)
   16 FORMAT(' BJ1'/(1X,6E11.4))
      U2=S*(0.,1.)+U2
      S=X*COS(P)
      V=-2./P*U1/U2*CMPLX(COS(S),-SIN(S))
      WRITE(21,26) V
   26 FORMAT(' V=',2E14.7)
      STOP
      END
```

```
C       LISTING OF THE INPUT DATA FILE MAUTZ3.DAT
C
 2010000 0.1570796E+01 0.8726646E-01
-0.3072582E+04 0.7368758E+04-0.6085100E+03 0.1710234E+02-0.2271001E+00
 0.1600171E-02-0.5961089E-05 0.9545773E-08 0.4163150E+05 0.3420211E+03
 0.1000000E+01-0.6024727E+04 0.1613512E+04-0.7532210E+02 0.1402590E+01
-0.1275602E-01 0.5832787E-04-0.1107698E-06 0.3072946E+05 0.2886431E+03
 0.1000000E+01 0.9999999E+00-0.1097659E-02 0.2461455E-04 0.1000000E+01
 0.1829893E-02-0.3191328E-04-0.1562498E-01 0.1427079E-03-0.5937434E-05
 0.4687498E-01-0.1998720E-03 0.7317495E-05
```

```
C        LISTING OF THE OUTPUT DATA FILE MAUTZ4.DAT
C
N1= 20, N2=10000, X= 0.1570796E+01, P= 0.8726646E-01
Y0
-0.3072582E+04 0.7368758E+04-0.6085100E+03 0.1710234E+02-0.2271001E+00
 0.1600171E-02-0.5961089E-05 0.9545773E-08 0.4163150E+05 0.3420211E+03
 0.1000000E+01-0.6024727E+04 0.1613512E+04-0.7532210E+02 0.1402590E+01
-0.1275602E-01 0.5832787E-04-0.1107698E-06 0.3072946E+05 0.2886431E+03
 0.1000000E+01 0.9999999E+00-0.1097659E-02 0.2461455E-04 0.1000000E+01
 0.1829893E-02-0.3191328E-04-0.1562498E-01 0.1427079E-03-0.5937434E-05
 0.4687498E-01-0.1998720E-03 0.7317495E-05
BJ
 0.4720014E+00 0.5668241E+00 0.2497016E+00 0.6903585E-01 0.1399603E-01
 0.2245355E-02 0.2983472E-03 0.3385059E-04 0.3352192E-05 0.2945642E-06
 0.2326610E-07 0.1669026E-08 0.1096726E-09 0.6648497E-11 0.3740814E-12
 0.1963744E-13 0.9661458E-15 0.4472615E-16 0.1955078E-17 0.8094814E-19
BY
 0.4100035E+00-0.3662806E+00-0.8763665E+00-0.1865369E+01-0.6248819E+01
-0.2995961E+02-0.1844800E+03-0.1379364E+04-0.1210935E+05-0.1219655E+06
-0.1385513E+07-0.1751893E+08-0.2439783E+09-0.3710196E+10-0.6116762E+11
-0.1086625E+13-0.2069184E+14-0.4204441E+15-0.9079853E+16-0.2076745E+18
BJ1(1)= 0.4359171E-01, BJ1(N2)=-0.2690578E-01
  N              U1                           U2
  0 0.3018775E+00 0.2622256E+00 0.3197845E+00 0.2777805E+00
  1 0.7036024E+00 0.8839002E+00 0.8676468E+00-0.7624704E-01
  2 0.5538185E+00 0.1409590E+01 0.1166432E+01-0.1124881E+01
  3 0.2884284E+00 0.1399768E+01 0.1236961E+01-0.3030578E+01
  4 0.2886049E+00 0.1320966E+01 0.1243170E+01-0.5803081E+01
  5 0.3049000E+00 0.1320967E+01 0.1243436E+01-0.9346033E+01
  6 0.3049000E+00 0.1323586E+01 0.1243443E+01-0.1358569E+02
  7 0.3045541E+00 0.1323586E+01 0.1243443E+01-0.1845940E+02
  8 0.3045541E+00 0.1323547E+01 0.1243443E+01-0.2390473E+02
  9 0.3045579E+00 0.1323547E+01 0.1243443E+01-0.2985677E+02
 10 0.3045579E+00 0.1323547E+01 0.1243443E+01-0.3624775E+02
 11 0.3045579E+00 0.1323547E+01 0.1243443E+01-0.4300740E+02
 12 0.3045579E+00 0.1323547E+01 0.1243443E+01-0.5006366E+02
 13 0.3045579E+00 0.1323547E+01 0.1243443E+01-0.5734352E+02
 14 0.3045579E+00 0.1323547E+01 0.1243443E+01-0.6477395E+02
 15 0.3045579E+00 0.1323547E+01 0.1243443E+01-0.7228288E+02
 16 0.3045579E+00 0.1323547E+01 0.1243443E+01-0.7980009E+02
 17 0.3045579E+00 0.1323547E+01 0.1243443E+01-0.8725819E+02
 18 0.3045579E+00 0.1323547E+01 0.1243443E+01-0.9459342E+02
 19 0.3045579E+00 0.1323547E+01 0.1243443E+01-0.1017465E+03
BJY(N1)=-0.1596493E-01, BJY(N2)=-0.3183099E-04
EJ1
-0.6917E+01-0.8147E+02-0.8840E+02-0.9237E+02-0.9453E+02-0.9555E+02
-0.9661E+02-0.9716E+02-0.9763E+02-0.9807E+02-0.9831E+02-0.9861E+02
-0.9881E+02-0.9898E+02-0.9916E+02-0.9927E+02-0.9940E+02-0.9951E+02
-0.9959E+02-0.9969E+02-0.9975E+02-0.9983E+02-0.9990E+02-0.9994E+02
-0.1000E+03-0.1000E+03-0.1001E+03-0.1001E+03-0.1002E+03-0.1002E+03
-0.1002E+03-0.1003E+03-0.1003E+03-0.1003E+03-0.1004E+03-0.1004E+03
-0.1004E+03-0.1004E+03-0.1004E+03-0.1005E+03-0.1005E+03-0.1005E+03
-0.1005E+03-0.1005E+03-0.1006E+03-0.1006E+03-0.1006E+03-0.1006E+03
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-0.1006E+03-0.1006E+03-0.1006E+03-0.1006E+03-0.1007E+03-0.1007E+03
-0.1007E+03-0.1007E+03-0.1007E+03-0.1007E+03-0.1007E+03-0.1007E+03
-0.1007E+03-0.1007E+03-0.1008E+03-0.1008E+03-0.1008E+03-0.1008E+03
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-0.1010E+03-0.1010E+03-0.1010E+03-0.1010E+03-0.1010E+03-0.1010E+03
-0.1010E+03-0.1010E+03-0.1010E+03-0.1010E+03-0.1010E+03-0.1010E+03
-0.1010E+03-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03
-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03
-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03
-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03
-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03
-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03-0.1011E+03
-0.1011E+03-0.1011E+03
V=-0.3443500E-01-0.1495431E+00
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